

Integrability and supersymmetry of Schrödinger-Pauli equations for neutral particles

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Abstract

Integrable quantum mechanical systems for neutral particles with spin $\frac{1}{2}$ and nontrivial dipole momentum are classified. It is demonstrated that such systems give rise to new exactly solvable problems of quantum mechanics with clear physical content. Solutions for three of them are given in explicit form. The related symmetry algebras and superalgebras are discussed. The presented classification is restricted to two-dimensional systems which admit matrix integrals of motion linear in momenta.

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I. INTRODUCTION

There are two inspiring notions in quantum mechanics called supersymmetry and superintegrability. Being formally independent, both of them are guide signs in searching for exactly solvable problems. Moreover, some of quantum mechanical systems, like the Hydrogen atom or isotropic harmonic oscillator, are both superintegrable and supersymmetric. Let us note that just such systems as a rule are very interesting and important.

A quantum mechanical system with n degrees of freedom is called superintegrable if it admits more than $n - 1$ integrals of motion. The system is treated as supersymmetric in two cases: when some of its integrals of motion form a superalgebra, and when its Hamiltonian has a specific symmetry with respect to the Darboux transform, called shape invariance [1].

The search for superintegrable systems was started with paper [2] where all the linear and quadratic integrals of motion for the 2d Schrödinger equation with an arbitrary potential had been presented. The systematic search for systems whose integrals of motion are first and second order polynomials in momenta was performed in [3], [4] and [5].

In last decades a number of new important results in this field has been obtained. In particular, integrable systems with second and third order integrals of motion in 2d and 3d Euclidean spaces had been intensively studied [6]-[9], the systems both electric and magnetic external fields and integrable systems with spin were discussed in [10] and [11], [12], [13].

However, there exist interesting integrable systems which were not studied systematically till now. They are neutral particles with non-trivial spin and dipole moment (e.g., neutrons) interacting with an external electromagnetic field. These particles are described by Schrödinger-Pauli equations which include the Stern-Gerlach or electric dipole terms. A perfect example of such system is the Pronko-Stroganov model [14] describing a neutron coupled to the field of the constant straight line current. This 2d system admits four integrals of motion (including Hamiltonian) [14], three of which are algebraically independent. Notice that there exist the relativistic [15] and arbitrary spin [16] versions of this system which are integrable and supersymmetric too [17].

In addition, the Pronko-Stroganov system is shape invariant and can be easily integrated using tools of supersymmetric quantum mechanics [18], [19]. This circumstance had motivated us to search for other supersymmetric Schrödinger-Pauli equations and classify matrix shape invariant potentials. In this way a number of new exactly solvable systems had been found [20],

[21].

In the present paper the integrable 2d Schrödinger-Pauli equations for neutral particles are classified. Like in paper [11] we restrict ourselves to the first order integrals of motion and classify all external fields which give rise to integrable and superintegrable systems. As a result a new class of integrable systems has been found. The majority of these systems is supersymmetric also, and one of them is shape invariant. Thus the germaneness between integrability and supersymmetry becomes apparent also in quantum mechanical models of neutral particles.

Integrals of motion of integrable systems presents powerful tools for finding their exact solutions. In this paper we restrict ourselves to solving three of the obtained systems. One of them appears to be a special case of models with shape invariant effective potentials that have recently been classified in [20] and [21]. The other system describes the neutron interacting with a periodic external field, which have both discrete and band energy spectra. One more system is rotationally invariant and includes a superposition of two external fields.

II. CLASSIFICATION PROBLEM

Let us consider a special class of Schrödinger-Pauli equations describing neutral fermions with non-trivial dipole momentum interacting with an external field. The corresponding stationary Schrödinger-Pauli equation looks as follows:

$$H\psi(\mathbf{x}) = E\psi(\mathbf{x}) \quad (1)$$

where

$$H = \left(\frac{p^2}{2m} + \frac{\lambda}{2m} \boldsymbol{\sigma} \cdot \mathbf{B} \right). \quad (2)$$

Here $\boldsymbol{\sigma}$ is the matrix vector whose components are Pauli matrices, $\mathbf{B} = \mathbf{B}(\mathbf{x})$ is a vector of magnetic field strength, and vector \mathbf{x} represents independent variables. In addition, λ denotes the constant of anomalous coupling which is usually represented as $\lambda = g\mu_0$ where μ_0 is the Bohr magneton and g is the Landé factor.

We shall classify integrable systems (1), i.e., find all Hamiltonians (2) admitting a sufficient number of integrals of motion which are linear combinations of momenta with *matrix* coefficients. In this paper we restrict ourselves to planar systems depending on two variables x_1 and

x_2 . However, bearing in mind possible generalizations of the presented results to 3d systems, we will not exclude the third coordinate a priori. Till an appropriate moment our analysis will be valid for both 2d and 3d systems.

To obtain more compact formulae let us rescall variables and reduce Hamiltonian (2) to the following form:

$$H = -\nabla^2 + \boldsymbol{\sigma} \cdot \mathbf{B} \quad (3)$$

where ∇ is the gradient vector with componets $\nabla_a = \frac{\partial}{\partial x_a}$. To achieve this goal it is sufficient to change in (2) $E \rightarrow \frac{1}{2m}E$ and $\mathbf{B} \rightarrow \frac{1}{\lambda}\mathbf{B}$.

For 3d systems $\nabla^2 = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}$ and \mathbf{B} depends on three variables x_1, x_2 , and x_3 . In the case of 2d systems the third variable x_3 and the corresponding derivatives should be deleted. This action can be formalized by imposing the constraint $\nabla_3\psi(\mathbf{x}) = 0$ and looking for integrals of motion which commute with H and ∇_3 .

Let us search for integrals of motion for hamiltonian (3) of the following generic form:

$$Q = \sigma^\mu (i\{\Lambda^{\mu a}, \nabla_a\} + \Omega^\mu) \quad (4)$$

where summation is imposed over the repeated indices $\mu = 0, 1, 2, 3$ and $a = 1, 2, 3$ or $a = 1, 2$ for 3d or 2d systems respectively, $\Lambda^{\mu a}$ and Ω^a are functions of \mathbf{x} , $\{\Lambda^{\mu a}, \nabla_a\} = \Lambda^{\mu a}\nabla_a + \nabla_a\Lambda^{\mu a}$, $\nabla_a = \frac{\partial}{\partial x_a}$, σ_μ are Pauli matrices:

$$\sigma^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

By definition, integrals of motion should commute with Hamiltonian,

$$[H, Q] \equiv HQ - QH = 0. \quad (5)$$

Substituting (3) and (4) into (5), using the relations

$$\{\sigma^a, \sigma^b\} = 2\delta^{ab}, \quad [\sigma^a, \sigma^b] = 2i\varepsilon^{abc}\sigma^c, \quad a, b, c = 1, 2, 3$$

and equating coefficients for linearly independent matrices and differential operators, we obtain the following system of determining equations for coefficients $\Lambda^{\mu a}$ and Ω^μ :

$$\Lambda_b^{\mu a} + \Lambda_a^{\mu b} = 0, \quad (6)$$

$$\Omega_a^0 = 0, \quad (7)$$

$$\Lambda^{ab} B_b^a = 0, \quad \Lambda^{0b} B_b^a = \varepsilon^{abc} \Omega^b B^c, \quad (8)$$

$$\Omega_b^a = 2\varepsilon^{acd} \Lambda^{cb} B^d. \quad (9)$$

Here the subindices denote derivatives w.r.t. the corresponding independent variables, i.e., $B_b^a = \frac{\partial B^a}{\partial x_b}$, etc., and summation is imposed over the repeated indices.

The system of equations (6)-(9) presents the necessary and sufficient conditions for commutativity of operators (3) and (4). Equations (6) and (7) are easily integrated, and their solutions have the following form:

$$\Lambda^{\mu a} = C^{\mu[ab]} x_b + C^{\mu a}, \quad \Omega^0 = C^0 \quad (10)$$

where $C^{\mu[ab]}$, $C^{\mu a}$ and C^0 are arbitrary constants satisfying the condition $C^{\mu[ab]} = -C^{\mu[ba]}$. Substituting (10) into (8) and (9) we obtain the overdetermined system of first order partial differential equations for functions Ω^a and B^a .

III. DETERMINING EQUATIONS AND EQUIVALENCE TRANSFORMATIONS FOR 2D SYSTEMS

The determining equations (6)-(9) and their partial solutions (10) are valid for both 2d and 3d equations (1). Starting with this point we restrict ourselves to two dimension systems depending on variables x_1 and x_2 . The corresponding function $\Lambda^{\mu a}$ in (10) and the related operator (4) are reduced to the following forms

$$\Lambda^{\mu a} = C^\mu \varepsilon^{ba} x_b + C^{\mu a}, \quad (11)$$

$$Q = \sigma^\mu (C^\mu L + C^{\mu a} P_a) + \sigma^a \Omega^a \quad (12)$$

where C^μ and $C^{\mu a}$ are arbitrary real constants, $a = 1, 2$, $b = 1, 2$, $\mu = 0, 1, 2, 3$, $P_a = -i\nabla_a$, $L = x_1 P_2 - x_2 P_1$, $\varepsilon^{ab} = -\varepsilon^{ba}$ and $\varepsilon^{12} = 1$. Moreover, the number of arbitrary constants in (11) and (12) can be reduced using the equivalence transformations which keep the form of Hamiltonian (3) up to multiplier $\frac{1}{\lambda^2}$:

$$x_a \rightarrow x_a + c_a, \quad (13)$$

$$x_a \rightarrow R_{ab}x_b, \quad a, b = 1, 2, \quad (14)$$

$$B^k \rightarrow \hat{R}^{kn}B^n, \quad \sigma^k \rightarrow \hat{R}^{kn}\sigma^n, \quad k, n = 1, 2, 3, \quad (15)$$

$$x_a \rightarrow \lambda x_a, \quad B^k \rightarrow \frac{1}{\lambda^2}B^k \quad (16)$$

where R_{ab}^1 and \hat{R}^{kn} are planar and spatial rotation matrices correspondingly, c_a and $\lambda \neq 0$ are real constants.

Let $C^0 = a \neq 0$ then up to shifts (13) we can set $C^{01} = C^{02} = 0$. Moreover, up to rotation transformations (15) we can restrict ourselves to $C^3 = b, C^1 = C^2 = 0$. Then, applying rotations (14) of variables x_1, x_2 and, if necessary, rotations (15) with $k, n = 1, 2$ we can reduce the remaining constants C^{ab} to $C^{11} = c_1, C^{22} = c_2, C^{12} = C^{21} = 0$ where c_1, c_2 and b are arbitrary parameters. As a result we reduce functions $\Lambda^{\mu a}$ in (11) to the following form:

$$\begin{aligned} \Lambda^{01} &= ax_2, \quad \Lambda^{02} = -ax_1, \quad \Lambda^{31} = bx_2 + d_1, \\ \Lambda^{32} &= -bx_1 + d_2, \quad \Lambda^{11} = c_1, \quad \Lambda^{22} = c_2 \end{aligned} \quad (17)$$

while all the other components of tensors $\Lambda^{\mu a}$ are zeros. In (17) we denote constants C^{31} and C^{32} as d_1 and d_2 correspondingly.

If one of parameters a or b (or both of them) are zero, we can use (13) again and reduce the set of functions (17) to the following non-trivial components:

$$\Lambda^{31} = bx_2, \quad \Lambda^{32} = -bx_1, \quad \Lambda^{11} = c_1, \quad \Lambda^{22} = c_2, \quad \Lambda^{01} = c_3, \quad \Lambda^{02} = c_4, \quad b \neq 0 \quad (18)$$

$$\Lambda^{01} = ax_2, \quad \Lambda^{02} = -ax_1, \quad \Lambda^{11} = c_1, \quad \Lambda^{22} = c_2, \quad a \neq 0, \quad (19)$$

$$\Lambda^{11} = c_1, \quad \Lambda^{22} = c_2, \quad \Lambda^{01} = c_3, \quad \Lambda^{02} = c_4. \quad (20)$$

The corresponding symmetry operators (12) look as follows:

$$Q = aL + \sigma_3 b \hat{L} + \sigma_1 c_1 P_1 + \sigma_2 c_2 P_2 + \sigma_a \Omega_a, \quad ab \neq 0, \quad (21)$$

$$Q = \sigma_3 b L + (c_3 + \sigma_1 c_1) P_1 + (c_4 + \sigma_2 c_2) P_2 + \sigma_a \Omega_a, \quad b \neq 0, \quad (22)$$

$$Q = aL + \sigma_1 c_1 P_1 + \sigma_2 c_2 P_2 + \sigma_a \Omega_a, \quad a \neq 0, \quad (23)$$

$$Q = (c_3 + \sigma_1 c_1) P_1 + (c_4 + \sigma_2 c_2) P_2 + \sigma_a \Omega_a \quad (24)$$

where $\hat{L} = \hat{x}_1 \hat{P}_2 - \hat{x}_2 \hat{P}_1$, $\hat{x}_1 = x_1 - \frac{d_2}{b}$, $\hat{x}_2 = x_2 + \frac{d_1}{b}$, $\hat{P}_a = -i \frac{\partial}{\partial \hat{x}_a}$. In addition, without loss of generality we can set $a = 1$ in (21) and (23) and $b = 1$ in (22).

Notice that for $c_1 = c_2 = c_3 = 0$ operators (23) and (24) are reduced to the following form:

$$Q = aL - \frac{k}{2} \sigma_3, \quad (25)$$

$$Q = c_3 P_1 + c_4 P_2 - \frac{n}{2} \sigma_3, \quad c_3 c_4 = 0 \quad (26)$$

where k , n , c_3 and c_4 are real parameters. Indeed, in accordance with (9), in this case functions Ω^a are reduced to constants, and hermitian matrix $\sigma_a \Omega^a$ is diagonalizable. Just operators (25) and (26) represent Lie symmetries which could be admitted by equation (1).

Equations (21)–(26) give representatives of the family of operators (12) defined up to equivalence transformations (13)–(16). In order these operators to be integrals of motion for Hamiltonian (3) functions B^1, B^2, B^3 and $\Omega^1, \Omega^2, \Omega^3$ have to satisfy equations (8) and (9) which, in view of (17)–(20), take the following form:

$$\begin{aligned} \Omega_1^3 &= -2c_1 B^2, \quad \Omega_2^3 = 2c_2 B^1, \\ \Omega_1^1 &= -2(bx_2 + d_1)B^3, \quad \Omega_2^2 = 2(d_2 - bx_1)B^1, \\ \Omega_1^2 &= 2c_1 B^3 + 2(bx_2 + d_1)B^1, \\ \Omega_2^1 &= -2c_2 B^3 + 2(bx_1 - d_2)B^1, \\ b(\hat{x}_1 B_2^3 - \hat{x}_2 B_1^3) &= c_1 B_1^1 + c_2 B_2^2, \\ a(x_1 B_2^1 - x_2 B_1^1) + \Omega^3 B^2 - \Omega^2 B^3 &= c_3 B_1^1 + c_4 B_2^1, \\ a(x_1 B_2^2 - x_2 B_1^2) + \Omega^1 B^3 - \Omega^3 B^1 &= c_3 B_1^2 + c_4 B_2^2, \\ a(x_1 B_2^3 - x_2 B_1^3) + \Omega^2 B^1 - \Omega^1 B^2 &= c_3 B_1^3 + c_4 B_2^3. \end{aligned} \quad (27)$$

where parameters d_1 , d_2 , c_3 , c_4 , a and b satisfy the conditions $ac_3 = ac_4 = 0$; $d_1 = d_2 = 0$ if $ab = 0$ and $a^2 + b^2 > 0$.

Thus the problem of classification of superintegrable planar Schrödinger-Pauli equations admitting first order integrals of motion is reduced to finding the general solution of determining equations (27). These equations should be solved consequently for all sets of parameters $a, b, d_1, d_2, c_1, c_2, c_3, c_4$ present in operators (21)–(26). Thus for operator of type (21) we set in (27) $c_3 = c_4 = 0$, $a \neq 0$, $b \neq 0$ while the remaining parameters are arbitrary, etc.

IV. LIE SYMMETRIES

Let us start with an important class of symmetry operators which are generators of continuous groups. They are the special cases of operators (12) corresponding to $\Lambda^{\mu a} = 0$ for all $\mu \neq 0$. Up to equivalence all such operators are given by equations (25) and (26). To classify the external fields $\mathbf{B} = (B_1, B_2, B_3)$ such that Hamiltonian (3) commutes with operators (25)

and (26) it is sufficient to solve equations (27) with $\Omega_1 = \Omega_2 = b = d_1 = d_2 = 0, \Omega_3 = \frac{n}{2}$ or $\Omega_3 = \frac{k}{2}$, while parameters c_3, c_4, n, k and a should satisfy the conditions $a(c_3^2 + c_4^2) = 0$ and $nk = 0$. As a result we obtain the list of the external fields and the corresponding symmetries presented in Table 1. A constant external field is not presented here since the corresponding Hamiltonian can be reduced to a direct sum of two Hamiltonians with trivial interaction terms.

Table 1. External fields with Lie symmetries

No	External field	Symmetry operators	Comments
1.	$B^1 = \cos(k\theta)f_1(r) + \sin(k\theta)f_2(r),$ $B^2 = \cos(k\theta)f_2(r) - \sin(k\theta)f_1(r),$ $B^3 = f_3(r)$	$\tilde{Q}_1 = L + \frac{k}{2}\sigma_3$	$\nabla \cdot \mathbf{B} \neq 0$
2.	$B^1 = \mu \cos(k\theta)r^{-k}, \quad B^2 = \mu \sin(k\theta)r^{-k},$ $B^3 = f_3(r)$	\tilde{Q}_1	
3.	$B^1 = \mu \cos(\theta)f_1(r), \quad B^2 = \mu \sin(\theta)f_1(r),$ $B^3 = f_2(r)$	$Q_1 = L + \frac{1}{2}\sigma_3$	
4.	$B^1 = \cos(\delta x_1)f_1(x_2) + \sin(\delta x_1)f_2(x_2),$ $B^2 = \cos(\delta x_1)f_2(x_2) - \sin(\delta x_1)f_1(x_2),$ $B^3 = f_3(x_2)$	$\tilde{Q}_2 = P_1 - \frac{\delta}{2}\sigma_3$	$\nabla \cdot \mathbf{B} \neq 0$
5.	$B^1 = \mu \exp(-x_2) \cos x_1,$ $B^2 = -\mu \exp(-x_2) \sin x_1, \quad B^3 = f_3(x_2)$	$Q_2 = P_1 - \frac{1}{2}\sigma_3$	Eq. (30) is shape invariant
6.	$B^1 = B^2 = 0, \quad B^3 = f(\mathbf{x})$	σ_3	decoupled
7.	$B^1 = B^2 = 0, \quad B^3 = f(x_1)$	$P_2, \quad \sigma_3$	decoupled
8.	$B^1 = B^2 = 0, \quad B^3 = f(r)$	$L, \quad \sigma_3$	decoupled

Here

$$r = \sqrt{x_1^2 + x_2^2}, \quad \theta = \arctan \frac{x_2}{x_1}, \quad (28)$$

$f_1(\cdot), f_2(\cdot), f_3(\cdot)$ and $f(\cdot)$ are arbitrary functions, and $\delta = 0$ or $\delta = 1$.

For \mathbf{B} given in Items 1 and 4 to be single valued and \tilde{Q}_1 to generate finite rotations for arbitrary values of theta, parameter k must be an integer.

The fields \mathbf{B} presented in Items 2 and 3 of Table 1 are particular cases of the field presented in Item 1. We specify these particular cases since the corresponding vector \mathbf{B} is divergent-free,

i.e., satisfies the condition

$$\nabla \cdot \mathbf{B} = 0 \quad (29)$$

which is not valid for more general fields given in Item 1. This condition is necessary if we interpret \mathbf{B} as a vector of magnetic field strength. Analogously, the divergent-free external field presented in Item 5 is a particular case of the more general field given in Item 4.

In the cases enumerated in Items 6 – 8 Hamiltonian (3) is reduced to a direct sum of Hamiltonians with scalar potentials, and the corresponding equation (1) is reduced to the system of two decoupled equations. This fact is indicated in the fourth column of Table 1.

Thus we find Hamiltonians (3) whose integrals of motion are generators of Lie symmetries. One more external field for which (3) admits Lie symmetries (but also a more general symmetry equivalent to (24) with $c_1 = c_2 = c_4 = 0, c_3 \neq 0$) is presented in Item 1 of Table 2.

Let us note that all obtained Hamiltonians admit a straightforward generalization which keeps its symmetries. Namely, in addition to (3), we can consider a more general Hamiltonian

$$H = -\nabla^2 + \boldsymbol{\sigma} \cdot \mathbf{B} + \omega \mathbf{B}^2 \quad (30)$$

where ω is an additional coupling constant.

Setting in (30) $\omega = 0$ we obviously come to the previous Hamiltonians (3). However, for all external fields presented in Table 1 the generalized Hamiltonians (30) admit the same Lie symmetries as Hamiltonian (3).

One more generalization of Hamiltonian (3) which keeps its symmetries can be written as:

$$H = -\nabla^2 + \boldsymbol{\sigma} \cdot \mathbf{B} + V \quad (31)$$

where $V = V(r)$ and $V = V(x_2)$ for the cases presented in Items 1, 2, 8 and 3, 4, 7 correspondingly.

V. NON-LIE INTEGRALS OF MOTION

In this section we present integrals of motion which are not generators of Lie symmetries. Doing this we restrict ourselves to the first order integrals of motion whose general form is given by equation (4) where at least one of coefficients Λ^{ab} , $a = 1, 2, 3$, is nonzero. Such integrals of motion can be represented in the forms given by equations (21)–(24). Moreover, the latest equation should be supplemented by the condition $c_1^2 + c_2^2 \neq 0$.

To evaluate the corresponding external fields \mathbf{B} and functions Ω^a it is necessary to solve equations (27) where parameters $a, b, c_1, c_2, c_3, c_4, d_1, d_2$ should satisfy one of the following sets of conditions:

$$ab \neq 0, \quad c_3 = c_4 = 0, \quad (32)$$

$$a = 0, \quad b \neq 0, \quad d_1 = d_2 = 0, \quad (33)$$

$$a \neq 0, \quad b = 0, \quad c_3 = c_4 = d_1 = d_2 = 0, \quad (34)$$

$$a = b = 0, \quad c_1^2 + c_2^2 \neq 0. \quad (35)$$

Moreover, conditions (32), (33), (34) and (35) correspond to symmetries (21), (33), (23) and (24) respectively. In addition, equivalence transformations (15) can be used to simplify the form of Q with fixed $\Lambda^{\mu a}$ and Ω^a .

Solving equations (27) under conditions (32)–(35) we obtain the list of external fields and the related integrals of motion which is presented in Table 2.

Table 2. External fields and higher symmetries

No	External field	Symmetry operators	Comments
1.	$B^1 = \mu \cos x_1, \quad B^2 = \mu \sin x_1,$ $B^3 = \nu$	$Q_2 = P_1 - \frac{1}{2}\sigma_3, \quad P_2,$ $Q_3 = \sigma_3(P_1 - \nu)$ $-\mu(\sigma_1 \cos x_1 + \sigma_2 \sin x_1)$	supersymmetric, see (42), (43)
2.	$B^1 = \frac{\mu k \sin(k\theta)}{r^2},$ $B^2 = -\frac{\mu k \cos(k\theta)}{r^2}, \quad B^3 = \frac{k\nu}{r^2}$	$\tilde{Q}_1 = L + \frac{k}{2}\sigma_3,$ $Q_4 = \sigma_3(\tilde{Q}_1 + \nu)$ $-\mu(\sigma_1 \sin(k\theta) - \sigma_2 \cos(k\theta))$	conformal, see (44); single valued for integer k
3.	$B^1 = \frac{\mu^2 x_2}{2\sqrt{\nu^2 - \mu^2 r^2}},$ $B^2 = -\frac{\mu^2 x_1}{2\sqrt{\nu^2 - \mu^2 r^2}}, \quad B^3 = \frac{\mu}{2}$	$Q_1 = L + \frac{1}{2}\sigma_3,$ $Q_5 = \sigma_1 P_1 + \sigma_2 P_2 - \frac{\mu}{2}(\sigma_1 x_2$ $-\sigma_2 x_1) - \frac{1}{2}\sigma_3 \sqrt{\nu^2 - \mu^2 r^2}$	supersymmetric, see (40), (41)
4.	$B^1 = \frac{x_2 \varphi'}{r},$ $B^2 = -\frac{x_1 \varphi'}{r}, \quad B^3 = -\mu(r\varphi)'$	$Q_1,$ $Q_6 = \sigma_1 P_1 + \sigma_2 P_2 + \mu(\sigma_3 Q_1$ $+ \sigma_1 x_2 \varphi - \sigma_2 x_1 \varphi) + \sigma_3 (\varphi + \nu)$	supersymmetric, see (37), (38), (39)

Here $\varphi = \varphi(r)$ is a solution of the following algebraic equation

$$(\mu^2 r^2 + 1) \varphi^2 + 2\nu \varphi = c, \quad (36)$$

μ , ν , k and c are real parameters. In particular, for $\nu = 0$, $c = \omega^2$ and $c = 0$, $\nu = -4\omega$ we obtain the following versions of the field presented in Item 4:

$$B^1 = -\frac{\omega \sin \theta \sinh \rho}{\cosh^3 \rho}, \quad B^2 = \frac{\omega \cos \theta \sinh \rho}{\cosh^3 \rho}, \quad B^3 = \frac{\omega}{\cosh^3 \rho}$$

and

$$B^1 = -\frac{\omega \sin \theta \sinh \rho}{\cosh^4 \rho}, \quad B^2 = \frac{\omega \cos \theta \sinh \rho}{\cosh^4 \rho}, \quad B^3 = \frac{\omega}{\cosh^4 \rho}$$

correspondingly, where we denote $r = \sinh \rho / \mu$.

Thus we have found the complete set of integrable planar models of neutral particles with Pauli interaction. The Hamiltonians of these models are given by equation (3) where \mathbf{B} is the vector of external field whose components are presented in Tables 1 and 2.

VI. ALGEBRAS AND SUPERALGEBRAS OF SYMMETRY OPERATORS

Symmetry operators presented in any item of Table 1 and Table 2 commute each other. In other words, the presented sets of integrals of motion together together with the corresponding Hamiltonians form bases of Abelian Lie algebras.

In addition, integrals of motion collected in Table 2 form interesting superalgebraic structures. Namely, operators presented in Item 4 satisfy the following relations:

$$Q_6^2 = \mathcal{H}, \quad [Q_6, \mathcal{H}] = 0, \quad (37)$$

$$[Q_1, \mathcal{H}] = [Q_1, Q_2] = 0. \quad (38)$$

where

$$\mathcal{H} = H + (\mu Q_1 + \nu)^2 + c \quad (39)$$

and H is the corresponding Hamiltonian (3). In other words, operators Q_1 , Q_2 and \mathcal{H} form a basis of the Lie superalgebra whose odd and even basis elements are Q_2 and $\langle Q_1, \mathcal{H} \rangle$ correspondingly. Relations (37) specify $N = 1$ SUSY.

Relations (37) and (38) can be effectively used to find eigenvectors and eigenvalues of the corresponding Hamiltonian (3). Indeed, the commuting hermitian operators Q_1 , Q_2 and H have common eigenvectors. To find eigenvectors for the first order differential operators Q_1 and Q_2 is much more easier than for the Hamiltonian which is a differential operator of the

second order. In addition, relation (37) makes it possible to find eigenvalues of H algebraically starting with eigenvalues for Q_1 and Q_2 .

Operators presented in Item 3 of Table 2 together with the corresponding Hamiltonian (3) also form a basis of superalgebra since the following relations are satisfied:

$$\begin{aligned} Q_5^2 &= \hat{\mathcal{H}}, \quad [Q_5, \hat{\mathcal{H}}] = 0, \\ [Q_1, \hat{\mathcal{H}}] &= [Q_1, Q_5] = 0 \end{aligned} \tag{40}$$

where

$$\hat{\mathcal{H}} = H + \mu Q_1. \tag{41}$$

In addition to their mutual commutativity the symmetry operators presented in Item 1 of Table 2 satisfy the following quadratic relations

$$Q_3^2 = H + 2\nu Q_2 + \nu^2, \tag{42}$$

$$\left(Q_3 - \frac{1}{2}\right)^2 = (Q_2 + \nu)^2. \tag{43}$$

Thus the corresponding models also admit $N = 1$ SUSY and can be effectively integrated.

Integrals of motion presented in Item 2 of Table 2 satisfy the following algebraic relations:

$$Q_4^2 = \tilde{Q}_1^2 + 2\nu\tilde{Q}_1 + \mu^2 + \nu^2 \tag{44}$$

which can be used to find eigenvalues of Q_2 using (well known) eigenvalues of Q_1 . Moreover, these integrals of motion together with operators $D = x_1 P_1 + x_2 P_2$, $K = r^2/2$ and the corresponding Hamiltonian H (3) form a basis of the five-dimensional Lie algebra since the following commutation relations are satisfied:

$$[H, D] = -2iH, \quad [K, D] = 2iK, \quad [K, H] = iD \tag{45}$$

while all the other commutators are trivial.

Relations (45) characterize conformal algebra $so(1,2)$, thus we deal with a model of conformal quantum mechanics (for definitions see, e.g., [22]).

VII. EXACT SOLUTIONS

In this section we use symmetries of the models found above to construct their exact solutions. Thanks to the presence of arbitrary parameters the number of qualitatively different models is too large to be considered in one paper, and it is the reason why we restrict ourselves only to three particular examples.

A. Neutron in periodic magnetic field

Let us start with the relatively simple Hamiltonian (3) with the components of magnetic field \mathbf{B} given in Item 1 of Table 2. Since this magnetic field depends on one spatial variable, it is reasonable to restrict ourselves to the one-dimensional eigenvalue problem

$$H\psi \equiv \left(-\frac{\partial^2}{\partial y^2} + \mu(\sigma_1 \cos y + \sigma_2 \sin y) + \nu\sigma_3 \right) \psi = E\psi \quad (46)$$

where we denote $x_1 = y$.

Hamiltonian in (46) includes periodic potential, thus this equation is a certain analog of the Bloch problem for electron. However, the considered eigenvalue problem is related to a neutral particle and includes a spin dependent potential.

Hamiltonian (46) commutes with operators Q_2 and Q_3 . Thus we can search for common eigenfunctions for Q_2 , Q_3 and H .

The eigenvalue problem for the first order differential operator Q_3 :

$$Q_3\psi_k \equiv (\sigma_3(P_1 - \nu) - \mu(\sigma_1 \cos(y) + \sigma_2 \sin(y))) \psi = k\psi_k \quad (47)$$

is easily solvable. The general solution for equation (47) is the two component function

$$\psi_k = \begin{pmatrix} \varphi_1(y) \\ \varphi_2(y) \end{pmatrix} \quad (48)$$

where

$$\begin{aligned} \phi_1(y) &= \exp\left(\frac{i(2\nu+1)y}{2}\right) ((C_1 k_- + C_2 \lambda_k) \cos(\lambda_k y) + (C_1 \lambda_k + C_2 k_-) \sin(\lambda_k y)), \\ \phi_2(y) &= -\mu \exp\left(\frac{i(2\nu-1)y}{2}\right) (C_1 \cos(\lambda_k y) + C_2 \sin(\lambda_k y)). \end{aligned} \quad (49)$$

Here $k_- = k - \frac{1}{2}$, $\lambda_k = \sqrt{k_-^2 - \mu^2}$, and the latest quantity must be real if we ask for solutions whose norm does not turn to infinity with growing y . In other words, admissible values of k are restricted by the condition

$$\left(k - \frac{1}{2}\right)^2 > \mu^2. \quad (50)$$

Let us present the admissible values of k more explicitly. First we note that up to the unitary transformation $H \rightarrow \sigma_3 H \sigma_3$ we can restrict ourselves to $\mu > 0$. Then, in accordance with (50)

there are two possibilities:

$$k \geq \mu + \frac{1}{2} \quad \text{or} \quad k \leq -\mu + \frac{1}{2}. \quad (51)$$

Since operator Q_2 commutes with H , eigenfunctions (48), (49) solve also equation (46). The corresponding eigenvalues E are easily calculated using algebraic relation $Q_2^2 = H$:

$$E = k^2. \quad (52)$$

In accordance with (51), (52) the admissible values of E are restricted by the following conditions:

$$\begin{aligned} E &\geq \left(\mu - \frac{1}{2}\right)^2 \quad \text{if} \quad \mu > \frac{1}{2}, \quad k < \frac{1}{2} - \mu, \\ E &\geq 0 \quad \text{if} \quad 0 < \mu \leq \frac{1}{2}, \quad k \leq \frac{1}{2} - \mu, \\ E &\geq \left(\mu + \frac{1}{2}\right)^2 \quad \text{if} \quad k \geq \mu + \frac{1}{2}. \end{aligned} \quad (53)$$

The probability density corresponding to solutions (49), i.e.,

$$\phi_1 \phi_1^* + \phi_2 \phi_2^* = (C_1^2 + C_2^2)k_-^2 + 2k_- \lambda_k C_1 C_2 + \frac{1}{4}(C_1^2 - C_2^2)\mu^2 \cos(2\lambda_k y) \quad (54)$$

is a periodic function. However its period can differ from the shift which keeps equation (46) invariant, i.e., from 2π . Such situation looks rather nonphysically, since the probability density calculated in a fixed frame of reference can differ from the density calculated in the equivalent frame of references shifted by 2π , in spite of that the equation (46) in these frames has exactly the same form.

There are two ways to obtain solutions whose amplitude is a periodic function with the period 2π . First it is possible to choose in (49)

$$C_1 = \pm C_2 = \frac{1}{2\sqrt{\pi k_- (k_- \pm \lambda_k)}} \quad (55)$$

and obtain solutions normalized at any invariance interval $[y, y + 2\pi]$:

$$\begin{aligned} \phi_1(y) &= \frac{1}{2} \sqrt{\frac{k_- \pm \lambda_k}{\pi k_-}} \exp\left(\frac{i(1 - 2\nu \pm 2\lambda_k)y}{2}\right), \\ \phi_2(y) &= \frac{\mu}{2\sqrt{\pi k_- (k_- \pm \lambda_k)}} \exp\left(-\frac{i(2\nu + 1 \mp 2\lambda_k)y}{2}\right). \end{aligned}$$

In this case there are no restrictions on eigenvalues E additional to (53).

The second way is to impose the following condition on the spectral parameter k :

$$k = \frac{1}{2}(\varepsilon\sqrt{n^2 + 4\mu^2} + 1) \quad (56)$$

where $n = 0, 1, 2, \dots$, $\varepsilon = \pm 1$. In this case the energy levels (52) are discrete and there are two branches: E_+ and E_- where

$$E_{\pm} = \frac{1}{4}(n^2 + 4\mu^2 \pm 2\sqrt{n^2 + 4\mu^2} + 1). \quad (57)$$

The corresponding eigenfunctions can be obtained from (48), (49) changing $\lambda_k \rightarrow \frac{n}{2}$ and using expression (56) for k . To obtain normalized solutions, arbitrary constants C_1 and C_2 should be restricted by the following condition:

$$(C_1^2 + C_2^2)(n^2 + 4\mu^2) + 2\varepsilon n C_1 C_2 \sqrt{n^2 + 4\mu^2} = \frac{2}{\pi}.$$

Thus like in the Bloch problem for electron [23] the energies of neutron moving in the periodic field represented in Item 1 of Table 2 can be continuous and have a band structure, see (53). In addition, there are solutions with discrete spectrum (57).

B. Rotationally invariant system

The next system which we consider includes the following Hamiltonian:

$$H = -\nabla^2 + \frac{\mu}{r^3}(\sigma_1 x_2 - \sigma_2 x_1) + \frac{\alpha}{r} \quad (58)$$

where μ and α are real parameters.

If $\alpha = 0$ Hamiltonian (58) coincides with operator (3) where B^1 and B^2 are components of the external field presented in Item 2 of Table 2 with $k = 1$ and $\nu = 0$. For nonzero α this is a Hamiltonian of type (31).

The additional term $\frac{\alpha}{x}$ which we include to obtain a more general model does not break the commutativity of the Hamiltonian with operators Q_1 and Q_4 . Using this fact and taking into account relations (44) we can expand solutions of the eigenvalue problem (1) for Hamiltonian (58) via eigenvectors of these operators satisfying

$$\begin{aligned} \left(L + \frac{\sigma_3}{2}\right) \psi_{k,\varepsilon} &= k \psi_{k,\varepsilon}, \quad k = \pm \frac{1}{2}, \pm \frac{3}{2}, \dots, \\ \left(\sigma_3 L - \frac{\mu}{r}(\sigma_1 x_2 - \sigma_2 x_1) + \frac{1}{2}\right) \psi_{k,\varepsilon} &= \varepsilon \sqrt{k^2 + \mu^2} \psi_{k,\varepsilon}, \quad \varepsilon = \pm 1. \end{aligned} \quad (59)$$

Solutions of equations (59) can be represented in the following form

$$\psi_{k,\varepsilon} = \frac{C_{k\varepsilon}\phi(r)}{\sqrt{r}} \begin{pmatrix} \exp(i(k - \frac{1}{2})\theta) \left(k + \varepsilon\sqrt{k^2 + \mu^2}\right) \\ i \exp(i(k + \frac{1}{2})\theta)\mu \end{pmatrix} \quad (60)$$

where $C_{k\varepsilon}$ are integration constants and polar coordinates (28) are used. Substituting (58), (60) and (28) into (1) we obtain the following ordinary differential equation for radial function $\phi = \phi(r)$:

$$\left(-\frac{\partial^2}{\partial r^2} + \frac{k^2 - \varepsilon\sqrt{k^2 + \mu^2}}{r^2} - \frac{\alpha}{r} \right) \phi = E\phi. \quad (61)$$

Let $\alpha = 0$ then equation (61) with the boundary condition

$$\phi = 0 \quad \text{if} \quad r = 0 \quad (62)$$

defines the eigenvalue problem of one dimensional conformal quantum mechanics. Its solutions can be expressed as follows:

$$E = p^2 > 0, \quad \phi = \sqrt{r} J_\nu(pr) \quad (63)$$

where $J_\nu(pr)$ is the Bessel function with

$$\nu = \frac{1}{2} \sqrt{1 + 4k^2 - 4\varepsilon\sqrt{k^2 + \mu^2}}. \quad (64)$$

If $\varepsilon = 1$ then admissible values of parameters μ and k are constrained by the following relation:

$$\left(k^2 - \frac{1}{4}\right)^2 \geq \mu^2 \quad \text{if} \quad \varepsilon = 1. \quad (65)$$

Let $\alpha > 0$ then solutions of equation (61) can be found using its analogy with the radial equation for Hydrogen atom. The eigenvalues E which correspond to square integrable solutions vanishing at the singularity point $r = 0$ look as follows:

$$E = -\frac{\alpha^2}{4 \left(n + \frac{3}{4} + \nu\right)^2} \quad (66)$$

where ν is parameter given in (64), n is a natural number and k satisfies conditions (59) and (65). The corresponding eigenvector ϕ can be expressed via a linear combination of Whittaker functions $M(a, b, x)$ and $W(a, b, x)$:

$$\phi = C_1 M\left(a, b, \frac{\alpha y}{a}\right) + C_2 W\left(a, b, \frac{\alpha y}{a}\right) \quad (67)$$

where

$$a = n + \nu + \frac{3}{4}, \quad b = \nu + \frac{1}{4}.$$

Notice that in contrast with the Hydrogen atom there is not a degeneration w.r.t. orbital quantum number. However, any energy level is infinitely degenerated since the corresponding eigenvector (67) includes two integration constants, and only one of them can be fixed by normalizing the wave function.

C. Shape invariant system

Let us consider the eigenvalue problem for Hamiltonian (30) where \mathbf{B} is the magnetic field whose components are given in Item 4 of Table 1 where $f_3 = 0$:

$$H\psi \equiv \left(-\nabla^2 + \lambda(1 - 2\kappa) \exp(-x_2)(\sigma_1 \cos x_1 - \sigma_2 \sin x_1) + \lambda^2 \exp(-2x_2)\right) \psi = E\psi. \quad (68)$$

Here λ is the integrated coupling constant and new parameter κ is introduced such that $\mu = \lambda(1 - 2\kappa)$.

Hamiltonian H in (68) admits integral of motion $Q_2 = P_1 - \frac{\sigma_3}{2}$. Thus it is possible to expand solutions of (68) via eigenvectors of Q_2 which look as follows:

$$\psi_p = \begin{pmatrix} \exp(i(p + \frac{1}{2})x_1)\varphi(x_2) \\ \exp(i(p - \frac{1}{2})x_1)\xi(x_2) \end{pmatrix} \quad (69)$$

and satisfy the condition $Q\psi_p = p\psi_p$.

Substituting (69) into (68) we come to the following equation:

$$\left(-\frac{\partial^2}{\partial y^2} + V_\kappa\right) \Phi = \varepsilon \Phi \quad (70)$$

where we denote

$$y = x_2, \quad \varepsilon = E - p^2 - \frac{1}{4}, \quad \Phi = \begin{pmatrix} \varphi \\ \xi \end{pmatrix} \quad (71)$$

and

$$V_\kappa = \lambda^2 \exp(-2y) - \lambda(2\kappa - 1) \exp(-y)\sigma_1 - p\sigma_3. \quad (72)$$

We can restrict ourselves to non-positive p , then solutions for $p > 0$ could be obtained changing $\Phi \rightarrow \sigma_1 \Phi$.

Let us consider the eigenvalue problem (70) with the following conditions:

$$y \geq 0, \quad \Phi(0) = 0, \quad \int_0^\infty \Phi^*(y)\Phi(y)dy < \infty.$$

Potential (72) belongs to the list of shape invariant matrix potentials found in [20], see equation (5.12) therein. It can be represented in the form $V = W_\kappa^2 - W'_\kappa + c_\kappa$ where

$$W_\kappa = -\kappa + \lambda \exp(-y)\sigma_1 - \frac{p}{2\kappa}\sigma_3, \quad c_\kappa = \kappa^2 + \frac{p^2}{4\kappa^2}. \quad (73)$$

The shape invariance means [1] that the superpartner potential $V_\kappa^+ = W_\kappa^2 + W'_\kappa + c_\kappa$ is equal to the initial potential with shifted parameter κ up to a constant term. And it is the case for potential (72) since

$$W_\kappa^2 + W'_\kappa = W_{\kappa+1}^2 - W'_{\kappa+1} + c_\kappa - c_{\kappa+1}.$$

Using the shape invariance it is possible to integrate equation (68) in a simple and straightforward way with using tools of SUSY quantum mechanics [20]. The eigenvalues ε and the corresponding state vectors are enumerated by natural numbers $n = 0, 1, \dots$. The ground state vector $\Phi_0(\kappa, y) = \begin{pmatrix} \varphi_0 \\ \xi_0 \end{pmatrix}$ should solve the equation

$$a_\kappa^- \Phi_0(\kappa, y) \equiv \left(\frac{\partial}{\partial x} + W_\kappa \right) \Phi_0(\kappa, y) = 0, \quad (74)$$

thus

$$\varphi_0 = z^{\frac{1}{2}-\kappa} K_{\nu+1}(z), \quad \xi_0 = z^{\frac{1}{2}-\kappa} K_\nu(z) \quad (75)$$

where $K_\nu(z)$ is modified Bessel function, $\nu = \frac{p}{2\kappa} - \frac{1}{2}$ and $z = \lambda \exp(-y)$.

Solutions which correspond to n^{th} excited state can be calculated using the following relation:

$$\Phi_n(\kappa, y) = a_\kappa^+ a_{\kappa+1}^+ \cdots a_{\kappa+n-1}^+ \Phi_0(\kappa + n, y) \quad (76)$$

where $a_\kappa^+ = -\frac{\partial}{\partial y} + W_\kappa(y)$. Finally, the corresponding values of spectral parameter ε have the following form [20]

$$\varepsilon = -N^2 - \frac{p^2}{4N^2} \Rightarrow E = p^2 - N^2 - \frac{p^2}{4N^2} + \frac{1}{4} \quad (77)$$

where $N = \kappa + n$ and n is a natural number.

In accordance with (77) eigenvalues E are invariant w.r.t. the change $p \rightarrow -p$, thus eigenvectors corresponding to a chosen n are linear combinations of functions (69):

$$\Psi_n = C_1 \begin{pmatrix} \exp(i(p + \frac{1}{2})x_1)\varphi_n(x_2) \\ \exp(i(p - \frac{1}{2})x_1)\xi_n(x_2) \end{pmatrix} + C_2 \begin{pmatrix} \exp(i(-p + \frac{1}{2})x_1)\xi_n(x_2) \\ \exp(i(-p - \frac{1}{2})x_1)\varphi_n(x_2) \end{pmatrix}. \quad (78)$$

Let $C_1 C_2 \neq 0$ then, in order the norm of this function be invariant w.r.t. the shifts $x_1 \rightarrow x_1 + 2\pi$ like the Hamiltonian (68), it is necessary to impose the following condition:

$$p = \frac{2m+1}{2}, \quad m = 0, 1, \dots$$

For solutions (78) with $C_1 \equiv 0$ or $C_2 \equiv 0$ the spectral parameter p can be quantized by imposing the periodic boundary condition with an arbitrary period.

VIII. DISCUSSION

In the present paper planar Schrödinger-Pauli equations for neutral particles, which admit first order constants of motion, are classified. The collection of such equations appears to be rather rich and interesting. In particular, it includes supersymmetric systems belonging to $N = 1$ SUSY quantum mechanics, the system with shape invariant Hamiltonian (68) and the system with Hamiltonian (58) (were $\alpha = 0$), which is conformally invariant.

Any Hamiltonian (3) with the external fields presented in Table 2 admits two integrals of motion, and following [11] -[13] we can call the related 2d systems superintegrable. Notice that this terminology is rather conventional, since the additional (spin) degree of freedom is ignored. To fix this degree of freedom we need an additional involutive integral of motion like matrix σ_3 , which can extend the possible number of symmetries. Maybe it is more natural to say that a 2d system with spin 1/2 is "superintegrable" (integrable) if it has at least three (two) independent integrals of motion. These speculations can be justified by the example given in Item 6 of Table 1. The corresponding 2d system admits a constant of motion, but it is not convenient to call it "integrable" since the related Hamiltonian is a direct sum of two 2d "nonintegrable" Hamiltonians which do not admit constants of motion provided B_3 is an arbitrary function of x_1 and x_2 .

Quantum mechanical systems with a sufficiently large number of constants of motion are usually exactly solvable, and it is the case for the models classified in the above. We restrict ourselves to solving three of them in section 7. To solve the model whose Hamiltonian is given

by equation (68) we use its shape invariance, and this is the second direct application of results of paper [20] where matrix superpotentials were classified, to a d -dimension models with $d > 1$. The first application of these results to planar systems with arbitrary spin can be found in paper [17]).

A physically interesting subclass of the classified systems includes Hamiltonians (46) and (68) whose effective potentials are periodic functions. Such potentials simulate interaction of neutron with a crystal lattice.

Fundamental results concerning the motion of electron in periodic electromagnetic field was formulated long time ago by F. Bloch [23]. Our analysis of solutions (49) shows that the neutron interacting with a periodic magnetic field can have both continuous and discrete energy spectrum.

The presented classification of planar Hamiltonians (3) admitting first order integrals of motion is complete. Nevertheless it can be considered as an intermediate outcome. First it is interesting to study integrable planar systems with higher order integrals of motion. An example of such system is the Pronko-Stroganov model [14] which admits symmetry operators of second order. Then, our analysis can be extended to systems of type (30) with arbitrary \mathbf{B} and V . Finally, the 3d superintegrable systems for neutrons are also waiting for their classification. In other words, superintegrable systems with spin whose investigation was started with paper [11] belong to a promising research field.

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